Annals of Combinatorics 8 (2004) 463-471
0218-0006/04/040463-9
DOI 10.1007/s00026-004-0233-3

Annals of Combinatorics

$4n-10$

Andreas W.M. Dress[1*], Jack Koolen[2†], and Vincent Moulton[3‡]

[1]Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22-26, 04103 Leipzig Germany
dress@mathematik.uni-bielefeld.de

[2]Division of Applied Mathematics, KAIST, 373-1 Kusongdong, Yusongku, Daejon 305 701 Korea
jhk@amath.kaist.ac.kr

[3]The Linnaeus Centre for Bioinformatics, Uppsala University, Box 598, 751 24 Uppsala Sweden
vincent.moulton@lcb.uu.se

Received March 19, 2003

AMS Subject Classification: 05D05, 05A99, 92B99

Abstract. We show that the maximal number $K_2(n)$ of splits in a 2-*compatible* split system on an n-set is exactly $4n-10$, for every $n > 3$.

Since $K_2(n) = CF_3(n)/2$ where $CF_3(n)$ is the maximal number of members in any 3-*cross-free* collection of (proper) subsets of an n-set, this gives a definitive answer to a question raised in 1979 by A. Karzanov who asked whether $CF_3(n)$ is, as a function of n, of type $O(n)$.

Karzanov's question was answered first by P. Pevzner in 1987 who showed $K_2(n) \leq 6n$, a result that was improved by T. Fleiner in 1998 who showed $K_2(n) \leq 5n$.

The argument given in the paper below establishes that the even slightly stronger inequality $K_2(n) \leq 4n-10$ holds for every $n > 3$; the identity $K_2(n) = 4n-10$ ($n > 3$) then follows in conjunction with a result from a previous paper that implies $K_2(n) \geq 4n-10$.

In that paper, it was also mentioned that — in analogy to well known results regarding maximal *weakly compatible* split systems — 2-compatible split systems of maximal cardinality $4n-10$ should be expected to be *cyclic*. Luckily, our approach here permits us also to corroborate this expectation. As a consequence, it is now possible to generate all 2-compatible split systems on an n-set ($n > 3$) that have maximal cardinality $4n-10$ (or, equivalently, all 3-cross-free set systems that have maximal cardinality $8n-20$) in a straight forward, systematic manner.

Keywords: 3-cross-free set systems, k-crossing set systems, laminar sets, cuts, cut multicommodity flow problem, multiflow locking, split systems, 2-compatible split systems, (in)compatible splits, cyclic split systems, weakly compatible split systems

* This work carried out whilst the author was a Distinguished Visiting Professor at the Dept. of Chem. Eng., City College, CUNY, New York.

† This research was done while the author was at FSPM-Strukturbildungsprozesse, University of Bielefeld, Germany, and CWI, Amsterdam, Netherlands, and also during a visit to Mid Sweden University, with the partial support of The Swedish Resaerch Council.

‡ **Communicating author.** The author thanks the Swedish Research Council for its support.

1. Introduction

In this paper, we study a problem raised by A. Karzanov in 1979, viz. his conjecture that the cardinality of any 3-*cross-free* collection of subsets of an n-set is bounded linearly by n (cf. [4]).

As this conjecture can be rephrased very conveniently using the language of *split theory* (cf. [1]), we begin by briefly recalling some of the basic terminology of that theory: A *split* $S = \{A, B\}$ of a set X is a bipartition of X into two sets A, B; in particular, we have $B = \overline{A} := X - A$. We denote by $\mathbf{S}(X)$ the set of all splits of X; any subset $\mathbf{S}$ of $\mathbf{S}(X)$ is called a *split system* (defined on X). Two splits S, S' of a set X are called *compatible* if there exist subsets $A \in S$ and $A' \in S'$ with $A \cap A' = \emptyset$, otherwise S and S' are called *incompatible*. We call a split system $\mathbf{S} \subseteq \mathbf{S}(X)$ 2-*compatible* if it does not contain an *incompatible triple* of splits, that is, a subset of 3 pairwise incompatible splits. More generally, a split system $\mathbf{S} \subseteq \mathbf{S}(X)$ is defined to be *k-compatible* if it does not contain a subset of $k+1$ pairwise incompatible splits.

At the end of the seventies, due newly discovered results on multicommodity flow problems such as those appearing in [4,5], it became of increasing interest to determine upper bounds for the cardinality of a k-compatible split system $\mathbf{S} \subseteq \mathbf{S}(X)$, defined on an n-set X. It is well known that every maximal 1-compatible split system contains exactly $2n-3$ distinct splits, and it was observed by M. Lomonosov that

$$\#\mathbf{S} \leq n + \frac{kn}{2} + \frac{kn}{3} + \cdots + \frac{kn}{\lfloor n/2 \rfloor} < n(1 + k\log_2(n))$$

always holds for a k-compatible split system defined on an n-set (see [2] for the simple proof of this inequality).

A. Karzanov conjectured that there is some universal constant c so that $\#\mathbf{S} \leq cn$ holds for any 2-compatible split system $\mathbf{S}$ defined on an n-set X. In [6], P. Pevzner[§] showed $\#\mathbf{S} \leq 6n$, a result that was subsequently improved to $\#\mathbf{S} \leq 5n$ by T. Fleiner in [3]. All of these results were stated in terms of *3-cross-free families*; however, they can easily be translated into the terminology of split systems, and vice-versa.

In [2], a simple construction was given, for each n-set with $n > 3$, that yields maximal 2-compatible split systems $\mathbf{S}$ with $\#\mathbf{S} = 4n - 10$ that are, in addition, *cyclic* i.e. there exists a bijection

$$\phi\colon [n] := \{1, \ldots, n\} \to X \quad (n := \#X)$$

such that $\mathbf{S}$ is contained in the split system $\mathbf{S}(\phi)$ consisting of all splits S of the form

$$\{\{\phi(i), \phi(i+1), \ldots, \phi(j-1)\}, \overline{\{\phi(i), \phi(i+1), \ldots, \phi(j-1)\}}\},$$

with $1 \leq i < j \leq n$. It was also shown there that every maximal 2-compatible *cyclic* split system $\mathbf{S}$ can be constructed in this way and, hence, must have cardinality $4n-10$. Based on these observations, it was then conjectured that — in analogy to well known results regarding maximal weakly compatible split systems (cf. [1]) — *every*

[§] In [3], a possible flaw in Pevzner's argument — probably due to poor translation — is pointed out.

2-compatible split system of maximal cardinality should be of this form[¶]. And it was observed in this context that indeed — as a consequence of results obtained in [1] — this is true in case for $4 \le \#X \le 5$ thus "explaining" the formula $4n-10$ as the "linear extrapolation" of the values obtained for $n = 4$ and $n = 5$.

The following two theorems establish this conjecture from [2], thus providing — in conjunction with results from [2] — a definitive answer to A. Karzanov's question described above:

Theorem 1.1. *The cardinality of any 2-compatible split system defined on a set X of cardinality $\#X := n > 3$ is bounded by $4n-10$.*

Theorem 1.2. *Every 2-compatible split system $\mathbf{S}$ defined on a set X of cardinality $\#X := n > 3$ is cyclic provided it has maximal cardinality $\#\mathbf{S} = 4n-10$.*

Finally, we want to point out that the key fact on which the proof of Theorem 1.1 is based, is the (non-)existence of certain configurations of splits in maximal 2-compatible split-systems (see Section 3). The discovery of these configurations was motivated, in part, by the existence of similar configurations in 3-*compatible cyclic* split systems (cf. [2, Section 5]).

2. Preliminary Results

We begin this section by introducing some further notation from split theory. Suppose that X is a finite set. For every proper subset A of X, we denote by S_A the split $S_A := \{A, \overline{A}\}$, induced by A. Whenever a subset A of X consists of one element $x \in X$ only, we may also write 'x' instead of '$\{x\}$' as long as no confusion can arise. In particular, we will write $\overline{x}$ instead of $\overline{\{x\}}$ and $S_x = \{x, \overline{x}\}$ instead of $S_{\{x\}} = \{\{x\}, \overline{\{x\}}\}$, for every $x \in X$. For every element $x \in X$ and every split $S \in \mathbf{S}(X)$, we denote by $S(x)$ the subset, A or B, in S that contains x, and by $\overline{S}(x)$, we denote its complement $\overline{S}(x) := \overline{S(x)} = X - S(x)$.

Now, given an element $x \in X$, a pair of splits $S_1, S_2 \in \mathbf{S}(X)$ is called an *x-pair* if $S_1(x) \cap S_2(x) = \{x\}$ and $S_1(x) \cup S_2(x) = X$ hold. Note that every x-pair consists of two distinct compatible splits. Moreover, a pair S_1, S_2 of splits from $\mathbf{S}(X)$ forms an x-pair if and only if there exists a split $S = \{A, B\} \in \mathbf{S}(\overline{x}) = \mathbf{S}(X - x)$ — uniquely determined by S_1 and S_2 — so that

$$\{S_1, S_2\} = \widehat{S} := \{\{A \cup x, B\}, \{A, B \cup x\}\}$$

holds. Next, given a split system $\mathbf{S} \subseteq \mathbf{S}(X)$ and a subset Y of X, define the *restriction* $\mathbf{S}|_Y$ *of* $\mathbf{S}$ *onto* Y by

$$\mathbf{S}|_Y := \{\{A, B\} \in \mathbf{S}(Y) : \ \{A, B\} = \{C \cap Y, D \cap Y\} \text{ for some } \{C, D\} \in \mathbf{S}\},$$

and note that, for every $\mathbf{S} \subseteq \mathbf{S}(X)$ and $x \in X$, the number

$$\#(\mathbf{S} - \{S_x\}) - \#\mathbf{S}|_{\overline{x}}$$

[¶] A split system $\mathbf{S} \subseteq \mathbf{S}(X)$ is called *weakly compatible* if there exist no four elements $x_0, x_1, x_2, x_3 \in X$ and three splits $S_1, S_2, S_3 \in \mathbf{S}$ with "$S_i(x_0) = S_i(x_j) \iff i = j$" for all $i, j \in \{1, 2, 3\}$.

coincides exactly with the cardinality $p_x(\mathbf{S}) := \#\mathbf{S}^x$ of the set $\mathbf{S}^x$ consisting of all x-pairs in $\mathbf{S}$. This follows as $\{A-x, B-x\}$ is a split in $\mathbf{S}(\overline{x})$ for every split $S \in \mathbf{S}(X) - \{S_x\}$, and we have $\{A-x, B-x\} = \{A'-x, B'-x\}$ for two distinct splits $\{A,B\}, \{A',B'\}$ in $\mathbf{S}(X)$ if and only if these two splits form an x-pair.

Hence, to prove Theorem 1.1, all we need to show is that there exists, for every 2-compatible split system $\mathbf{S} \subseteq \mathbf{S}(X)$, some $x \in X$ with $p_x(\mathbf{S}) \leq 3$ because, by induction with respect to $\#X$, this would imply

$$\begin{aligned}\#\mathbf{S} &\leq \#\{S_x\} + \#\mathbf{S}|_{\overline{x}} + p_x(\mathbf{S}) \\ &\leq 1 + (4(\#X-1) - 10) + 3 \\ &= 4\#X - 10.\end{aligned}$$

Thus, in order to prove Theorem 1.1, we need to study x-pairs in 2-compatible split systems. To this end, we begin by noting that for any two splits $S, S' \in \mathbf{S}(X)$, the following assertions are equivalent:

(i) S and S' are incompatible,
(ii) $A \in S$ and $A' \in S'$ imply $A \cap A', \overline{A} \cap A', A \cap \overline{A'}, \overline{A} \cap \overline{A'} \neq \emptyset$,
(iii) $A \in S$ and $A' \in S'$ imply $A \cap A' \neq \emptyset$,
(iv) there is some $A \in S$ that is neither contained in — nor contains — any $A' \in S'$,
(v) no $A \in S$ is either contained in — or contains — any $A' \in S'$.

Moreover, defining two collections of splits $\mathbf{S}', \mathbf{S}'' \subseteq \mathbf{S}(X)$ to be *(in)compatible* if any two splits S', S'' with $S' \in \mathbf{S}'$ and $S'' \in \mathbf{S}''$ are (in)compatible, we observe

Lemma 2.1. *If $x \in X$, and U and V are two distinct splits from $\mathbf{S}(\overline{x})$, then the two corresponding x-pairs $\widehat{U}$ and $\widehat{V}$ are incompatible if U and V are incompatible while, otherwise, there exists exactly one pair U_1, V_1 of incompatible splits with $U_1 \in \widehat{U}$ and $V_1 \in \widehat{V}$.*

In particular, every split S in the restriction $\mathbf{S}|_{\overline{x}}$ of any 2-compatible split system $\mathbf{S}$ that contains the two x-pairs $\widehat{U}$ and $\widehat{V}$ must be compatible with U or with V.

Proof. The first statement is obvious. To prove the second, assume $U = \{A, B\}$, $V = \{A', B'\}$, and $A \cap A' = \emptyset$. Then, we have $\widehat{U} = \{\{A \cup x, B\}, \{A, B \cup x\}\}$ and $\widehat{V} = \{\{A' \cup x, B'\}, \{A', B' \cup x\}\}$ and, clearly, of these four splits only $\{A \cup x, B\}$ and $\{A' \cup x, B'\}$ are incompatible. ∎

As a consequence of this lemma, we see that two distinct x-pairs can never be compatible; if they are not incompatible, we call them *semi-compatible*.

Proposition 2.2. *Suppose that $\mathbf{S} \subseteq \mathbf{S}(X)$ is a 2-compatible split system, and that $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W} \subseteq \mathbf{S}$ are three distinct x-pairs for some $x \in X$. If $\mathcal{U}$ is semi-compatible with $\mathcal{V}$, and $\mathcal{V}$ is semi-compatible with $\mathcal{W}$, then $\mathcal{U}$ is semi-compatible with $\mathcal{W}$.*

Proof. Choose splits $U, V, W \in \mathbf{S}(\overline{x})$ with $\mathcal{U} = \widehat{U}$, $\mathcal{V} = \widehat{V}$, and $\mathcal{W} = \widehat{W}$, and assume that $\mathcal{U}$ and $\mathcal{W}$ are incompatible. Choose $A \in U$ and $A' \in V$ with $A \cap A' = \emptyset$, and choose $A'' \in W$ with $A'' \subseteq A'$ or $A' \subseteq A''$. Then, $A' \subseteq A''$ must hold because, otherwise, we would have $A'' \cap A \subseteq A' \cap A = \emptyset$. However, in this case, the three splits $S_{A \cup x} = \{A \cup x, \overline{A \cup x}\} \in \mathcal{U}$, $S_{A''} = \{A'', \overline{A''}\} \in \mathcal{W}$, and $S_{A' \cup x} = \{A' \cup x, \overline{A' \cup x}\} \in \mathcal{V}$ would form an incompatible triple of splits in $\mathbf{S}$. ∎

Proposition 2.2 in turn implies

Proposition 2.3. *If $\mathbf{S} \subseteq \mathbf{S}(X)$ is a 2-compatible split system and if $x \in X$ is any element of X, then either there exist exactly two x-pairs that are incompatible, or any two x-pairs are semi-compatible and there exists a sequence $S_1 = \{A_1, B_1\}, S_2 = \{A_2, B_2\}, \ldots, S_k = \{A_k, B_k\}$ of splits of $\overline{x}$ with $A_1 \subset A_2 \subset \cdots \subset A_k$ (and hence $B_1 \supset B_2 \supset \cdots \supset B_k$) so that*

$$\mathbf{S}^x = \{\widehat{S_i} : i = 1, \ldots, k\}$$

holds.

Proof. If $\mathcal{U}, \mathcal{V}$, and $\mathcal{W}$ were three distinct x-pairs from $\mathbf{S}$ so that $\mathcal{U}$ and $\mathcal{V}$ are incompatible, then either $\mathcal{V}$ and $\mathcal{W}$ or $\mathcal{U}$ and $\mathcal{W}$ would have to be incompatible too, in view of Proposition 2.2. So, assume that $\mathcal{V}$ and $\mathcal{W}$ were incompatible. Choose $U_1 \in \mathcal{U}$ and $W_1 \in \mathcal{W}$ according to Lemma 2.1 so that U_1 and W_1 are incompatible. Then, U_1, W_1, V would form an incompatible triple of splits in $\mathbf{S}$, for every split $V \in \mathcal{V}$, which is clearly a contradiction.

So, assume now that all x-pairs in $\mathbf{S}^x$ are semi-compatible. All we need to show then is that, given some $A \subseteq \overline{x}$ with S_A, $S_{A\cup x} \in \mathbf{S}^x$, the set $\mathcal{A}$ of subsets $A' \subseteq \overline{x}$ with $\widehat{S}_{A'} \in \mathbf{S}^x$ and $A' \subseteq A$ or $A \subseteq A'$ is linearly ordered. If not, there would exist $A_1, A_2 \in \mathcal{A}$ with $A_1 \cap A_2 \neq A_1, A_2$ and either $A_1, A_2 \subset A$ or $A \subset A_1, A_2$.

In the first case, we must have $A_1 \cap A_2 = \emptyset$, and $S_{A_1\cup x} = \{A_1 \cup x, \overline{A_1 \cup x}\}$, $S_{A_2\cup x} = \{A_2 \cup x, \overline{A_2 \cup x}\}$ and $S_A = \{A, \overline{A}\}$ would form an incompatible triple of splits in $\mathbf{S}$ because $S_{A_1\cup x}$ is clearly not compatible with $S_{A_2\cup x}$, and A neither contains — nor is contained in — either of $A_1 \cup x$, $\overline{A_1 \cup x}$, $A_2 \cup x$, or $\overline{A_2 \cup x}$.

In the second case, we must have $A_1 \cup A_2 = \overline{x}$, and $S_{A_1} = \{A_1, \overline{A_1}\}$, $S_{A_2} = \{A_2, \overline{A_2}\}$, and $S_{A\cup x} = \{A \cup x, \overline{A \cup x}\}$ would form an incompatible triple of splits in $\mathbf{S}$ because, now, S_{A_1} and S_{A_2} are incompatible and $A \cup x$ neither contains — nor is contained in — either of $A_1, \overline{A_1}, A_2$, or $\overline{A_2}$. ∎

It follows from this proposition and the discussion above that no 2-compatible split system $\mathbf{S} \subseteq \mathbf{S}(X)$ with $\#\mathbf{S} \geq 4\#X - 10$, $\#X \geq 5$, can contain a pair of incompatible x-pairs, for any $x \in X$ provided Theorem 1.1 holds for 2-compatible split systems defined on an n-set with $n < \#X$.

3. Proof of Theorem 1.1

It is now straight forward to establish Theorem 1.1 by induction noting that, for obvious reasons, it holds for $4 \leq \#X \leq 5$. So, assume $\#X > 5$ and that our theorem holds for 2-compatible split systems defined on an n-set with $n < \#X$. We consider quadruples x, A, B, C consisting of an element $x \in X$ and a partition $\{A, B, C\}$ of $\overline{x}$ into three disjoint, non-empty sets so that $S_{A\cup x}$, $S_{A\cup B} = S_{C\cup x}$, and $S_{A\cup B\cup x} = S_C$ all belong to $\mathbf{S}$. Clearly, such quadruples must exist in case $\#\mathbf{S} \geq 4\#X - 10$, because any pair $\widehat{\{A_1, B_1\}}$ and $\widehat{\{A_2, B_2\}}$ of semi-compatible x-pairs $\widehat{\{A_1, B_1\}}, \widehat{\{A_2, B_2\}} \in \mathbf{S}^x$ with, say, $A_1 \subset A_2$ gives rise to such a quadruple, viz. $x, A_1, A_2 - A_1, B_2$.

Among all such quadruples, choose one, say x_0, A_0, B_0, and C_0, for which C_0 is maximal.

Note — for further use in the next section — that $2\#C_0 \geq \#X - 1$ must hold in case $\#\mathbf{S} \geq 4\#X - 10$ because, given any triple $S_1 = \{A_1, B_1\}$, $S_2 = \{A_2, B_2\}$, and $S_3 = \{A_3, B_3\}$ of $\overline{x_0}$-splits with $A_1 \subset A_2 \subset A_3$ and $\widehat{S_1}, \widehat{S_2}, \widehat{S_3} \in \mathbf{S}^{x_0}$, A_2 as well as B_2 could serve as C_0 and, because $\#A_2 + \#B_2 = \#X - 1$, the larger one of these two candidates for C_0 must have cardinality at least $\frac{1}{2}(\#X - 1)$. So, the maximal C_0 must also have at least this cardinality that, in view of $\#X > 5$, is at least $2\frac{1}{2}$. So, $\#C_0 > 2$ must hold.

Next, consider an arbitrary element $a_0 \in A_0$. Choose $S_1 = \{A_1, B_1\}$, $S_2 = \{A_2, B_2\}$, $\ldots$, $S_k = \{A_k, B_k\}$ in $\mathbf{S}(\overline{a_0})$ with $A_1 \subset A_2 \subset \cdots \subset A_k$ and

$$\mathbf{S}^{a_0} = \{\widehat{S_i}:\ i = 1, \ldots, k\}.$$

Then all we need to establish is that $k \leq 3$ must hold.

To this end, we observe first that $B_i \cap (A_0 \cup B_0 \cup x_0) \neq \emptyset$ must hold for all $i = 1, \ldots, k$ because, otherwise, we would have $B_i \subseteq C_0$ and

$$A_0 \cup B_0 \cup x_0 = \overline{C_0} \subseteq \overline{B_i} = a_0 \cup A_i,$$

that is, $A_0 \subseteq a_0 \cup A_i$, $B_0 \subseteq A_i$, and $x_0 \in A_i$; so the three splits

$$S_{A_i} = \{A_i, B_i \cup a_0\},\ S_{A_0 \cup B_0} = \{A_0 \cup B_0, C_0 \cup x_0\},\ S_{A_0 \cup x_0} = \{A_0 \cup x_0, B_0 \cup C_0\}$$

would form an incompatible triple of splits in $\mathbf{S}$ in view of

$$B_0 \subseteq A_i \cap (A_0 \cup B_0),$$

$$x_0 \in A_i \cap (C_0 \cup x_0),$$

$$a_0 \in (B_i \cup a_0) \cap (A_0 \cup B_0),$$

$$B_i \subseteq (B_i \cup a_0) \cap (C_0 \cup x_0),$$

$$x_0 \in A_i \cap (A_0 \cup x_0),$$

$$B_0 \subseteq A_i \cap (B_0 \cup C_0),$$

$$a_0 \in (B_i \cup a_0) \cap (A_0 \cup x_0),$$

and

$$B_i \subseteq (B_i \cup a_0) \cap (B_0 \cup C_0).$$

So, we must have

$$B_i \cap (A_0 \cup B_0 \cup x_0) \neq \emptyset$$

for all $i = 1, \ldots, k$. By symmetry, we must also have

$$A_i \cap (A_0 \cup B_0 \cup x_0) \neq \emptyset$$

for all $i = 1, \ldots, k$.

Now, if $B_{k-1} \cap C_0 = \emptyset$ were to hold, the quadruple $a_0, B_k, B_{k-1} \cap A_k, A_{k-1}$ would form a quadruple of the kind considered above, with $C_0 \subseteq A_{k-1}$. So, by our choice of C_0, we must have $A_{k-1} = C_0$ or $B_{k-1} \cap C_0 \neq \emptyset$. However, if $A_{k-1} = C_0$, then $S_{A_{k-1} \cup a_0} = \{C_0 \cup$

$a_0, x_0 \cup (A_0 - a_0) \cup B_0\}$, $S_{C_0 \cup x_0} = \{C_0 \cup x_0, A_0 \cup B_0\}$, and $S_{A_0 \cup x_0} = \{A_0 \cup x_0, B_0 \cup C_0\}$ would form an incompatible triple of splits in $\mathbf{S}$. So, we must have $B_{k-1} \cap C_0 \neq \emptyset$ and, hence, $B_i \cap C_0 \neq \emptyset$ for all $i = 1, \ldots, k-1$. By symmetry, also $A_2 \cap C_0 \neq \emptyset$ must hold and, hence, $A_i \cap C_0 \neq \emptyset$ for all $i = 2, \ldots, k$.

Finally, we note that $i \in \{2, \ldots, k\}$ implies $A_{i-1} \cap C_0 = \emptyset$ or $B_i \cap C_0 = \emptyset$ because the three splits

$$S_{B_{i-1}} = S_{A_{i-1} \cup a_0} = \{A_{i-1} \cup a_0, B_{i-1}\},$$

$$S_{A_i} = S_{B_i \cup a_0} = \{A_i, B_i \cup a_0\},$$

and

$$S_{C_0} = \{C_0, A_0 \cup B_0 \cup x_0\}$$

would form an incompatible triple. In view of $A_2 \cap C_0 \neq \emptyset$ and $B_{k-1} \cap C_0 \neq \emptyset$, this implies $B_3 \cap C_0 = \emptyset$ as well as $A_{k-2} \cap C_0 = \emptyset$ and hence, in particular, $k \leq 3$ as claimed. ∎

4. Proof of Theorem 1.2

To establish Theorem 1.2, we also proceed by induction with respect to $\#X$, noting as above that — in view of the results obtained in [1] — Theorem 1.2 clearly holds in case $4 \leq \#X \leq 5$. So, using the notations and arguments from the previous section, we must have $k = 3$, $B_3 \cap C_0 = \emptyset$, and $A_1 \cap C_0 = \emptyset$. Moreover, assuming that — without loss of generality — $x_0 \in A_2$ holds, we may assume that — with $n := \#X - 1 \geq 5$ — we have $\overline{a_0} = [n] = \{1, \ldots, n\}$ and $\mathbf{S}' := \mathbf{S}|_{\overline{a_0}}$ is a subset of

$$\mathbf{S}^* := \{\{\{i, i+1, \ldots, j-1\}, [n] - \{i, i+1, \ldots, j-1\}\} : 1 \leq i < j \leq n\},$$

that is, it is a subset of the standard cyclic split system on $\{1, \ldots, n\}$.

In addition, we may assume that

$$x_0 = n,\ A_0 - a_0 = \{1, \ldots, i_0 - 1\},\ B_0 = \{i_0, \ldots, j_0 - 1\},\ C_0 = \{j_0, \ldots, n-1\},$$

and

$$B_2 = \{l_0, l_0 + 1, \ldots, k_0 - 1\}$$

holds for some $i_0, j_0, l_0, k_0 \in \{1, \ldots, n\}$ with $1 \leq i_0 < j_0 < n-1$ and $1 \leq l_0 < j_0 < k_0 < n$. And, using these notations, we must also have

$$A_1 \subseteq A_2 \cap \overline{C_0} = \{n\} \cup \{1, \ldots, l_0 - 1\}$$

and

$$B_3 \subseteq B_2 \cap \overline{C_0} = \{l_0, \ldots, j_0 - 1\}.$$

We now recall from [2] that every cyclic 2-compatible split system $\mathbf{S}' \subseteq \mathbf{S}^*$ with $\mathbf{S}' = 4n - 10$ contains all splits of the form $\{\{i, i+1\}, [n] - \{i, i+1\}\}$ $(i = 1, \ldots, n)$ and that every set C with $\{C, [n] - C\} \in \mathbf{S}'$ and $\#C \geq 3$ contains a subset C' with $\#C' = 3$ and $\{C', [n] - C'\} \in \mathbf{S}'$.

Consequently, there is some $m_0 \in C_0$ with $m_0 + 2 \leq n-1$ and the splits

$$\{\{m_0, m_0+1, m_0+2\}, [n] - \{m_0, m_0+1, m_0+2\}\},$$

$$\{\{m_0, m_0+1\}, [n] - \{m_0, m_0+1\}\},$$

and

$$\{\{m_0+1, m_0+2\}, [n] - \{m_0+1, m_0+2\}\}$$

are all contained in $\mathbf{S}'$. Clearly, the corresponding splits in $\mathbf{S}$ all must be of the form $S_{\{m_0,m_0+1,m_0+2\}}$, $S_{\{m_0,m_0+1\}}$, and $S_{\{m_0+1,m_0+2\}}$ because every split of the form $S_{C\cup a_0}$ ($\emptyset \subset C \subseteq C_0$) forms an incompatible triple of splits together with $S_{A_1\cup a_0}$ and $S_{B_3\cup a_0}$ in view of $B_3 \cap C_0 = \emptyset$ and $A_1 \cap C_0 = \emptyset$. This, however, provides us with a quadruple m_0, $\{m_0+1\}$, $\{m_0+2\}$, $X - \{m_0, m_0+1, m_0+2\}$ of the kind considered in the beginning of Section 3 and, hence, it implies $\#A_0 = \#B_0 = 1$, that is, $A_0 = \{a_0\}$, $i_0 = 1, j_0 = 2, l_0 = 1, A_1 = \{n\}$, and $B_0 = B_3 = \{1\}$.

With $a_0 := 0$, it is now easy to see that $\mathbf{S}$ must be contained in the cyclic split system

$$\{\{\{i, i+1, \ldots, j-1\}, \{0, \ldots, i-1\} \cup \{j, \ldots, n\}\} : 0 \leq i < j \leq n\}.$$

Indeed, both of the two extensions of $\{A_1, B_1\}$,$\{A_2, B_2\}$, and $\{A_3, B_3\}$ clearly belong to this split system in view of $n \in A_1 \subseteq A_2 \subseteq A_3$ and $1 \in B_3 \subseteq B_2 \subseteq B_1$. And the unique extension $S \in \mathbf{S}$ of every other split

$$\{\{i, i+1, \ldots, j-1\}, \{[n] - \{i, i+1, \ldots, j-1\}\} \quad (1 \leq i < j \leq n)$$

in $\mathbf{S}'$ must also be of this form: This is trivially so in case $i = 1$. And in case $i > 1$, we cannot have

$$\{\{0\} \cup \{i, i+1, \ldots, j-1\}, [n] - \{i, i+1, \ldots, j-1\}\} \in \mathbf{S}$$

because this split forms an incompatible triple together with

$$\{\{0, 1\}, \{2, \ldots, n\}\} = S_{A_0 \cup B_0}$$

and

$$\{\{1, \ldots, n-1\}, \{n, 0\}\} = S_{A_0 \cup x_0}$$

in view of $2 \leq i \leq n-1$,

$$0, i \in \{0\} \cup \{i, i+1, \ldots, j-1\},$$

and

$$1, n \in [n] - \{i, i+1, \ldots, j-1\}.$$

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Acknowledgments. The authors would like to thank both A. Karzanov and P. Pevzner for helpful comments and suggestions.

References

1. H.-J. Bandelt and A. Dress, A canonical decomposition theory for metrics on a finite set, Adv. Math. **92** (1992) 47–105.
2. A. Dress, M. Klucznik, J. Koolen, and V. Moulton, $\mathbf{2nk}-\binom{\mathbf{k+1}}{\mathbf{2}}$: A note on extremal combinatorics of cyclic split systems, Sém. Lothar. Combin. **47** (2001) B47b.
3. T. Fleiner, The size of 3-cross-free families, Combinatorica **21** (2001) 445–448.
4. A. Karzanov, Combinatorial methods to solve cut-determined multiflow problems, In: Combinatorial Methods for Flow Problems, no.3, A. Karzanov, Ed., Vsesoyuz. Nauchno-Issled. Inst. Sistem. Issled., Moscow, 1979, pp. 6–69 (in Russian).
5. A. Karzanov and P. Pevzner, A description of the class of cut-non-determined maximum multiflow problems, In: Combinatorial Methods for Flow Problems, no.3 A. Karzanov, Ed., Vsesoyuz. Nauchno-Issled. Inst. Sistem. Issled., Moscow, 1979, pp. 70–81 (in Russian).
6. P. Pevzner, Non-3-crossing families and multicommodity flows, Amer. Math. Soc. Transl. Ser. 2, **158** (1994) 201–206. (Translated from: P. Pevzner, Linearity of the cardinality of 3-cross free sets, In: Problems of Discrete Optimization and Methods for Their Solution A. Fridman, Ed., Moscow, 1987, pp. 136–142 (in Russian)).